VARIATIONS OF A CLASS OF MONGE-AMPÈRE TYPE FUNCTIONALS AND THEIR APPLICATIONS

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Abstract. In this paper, we study a class of Monge-Ampère type functionals arising from the L_p dual Minkowski problem in convex geometry. As an application, we obtain some existence and non-uniqueness results for the problem.

1. Introduction

The characterisation problem of geometric measures in convex geometry has a long history and strong influence on fully nonlinear PDEs. A best known example is the classical Minkowski problem. For a full discussion on this problem and its resolution, one may consult Cheng-Yau [18] and Pogorelov [34]. Other important geometric measures in Brunn-Minkowski theory include curvature measures and area measures, and the associated problems of prescribing curvature and area measures were also intensively studied. See Schneider's book [35] for a comprehensive introduction.

Most recently Lutwak-Yang-Zhang [33] introduced the L_p dual curvature measures and proposed the associated Minkowski type problem. Let $\mathcal{K}_0$ be the set of all convex bodies (i.e., compact convex sets that have non-empty interior) in $\mathbb{R}^{n+1}$ containing the origin in their interiors. Associated to each $\Omega \in \mathcal{K}_0$ are the support function $u = u_\Omega : \mathbb{S}^n \to \mathbb{R}$ and the radial function $r = r_\Omega : \mathbb{S}^n \to \mathbb{R}$, which are respectively defined by $u(x) = \max\{x \cdot z : \ z \in \Omega\}$, and $r(\xi) = \max\{\lambda : \ \lambda\xi \in \Omega\}$. Let $\vec{r}(\xi) = \vec{r}_\Omega(\xi) := r_\Omega(\xi)\xi$. Then $\partial\Omega = \{\vec{r}(\xi) : \ \xi \in \mathbb{S}^n\}$. Denote by $\nu = \nu_\Omega : \ \partial\Omega \to \mathbb{S}^n$ the spherical image, namely $\nu(z) = \{x \in \mathbb{S}^n : z \cdot x = u_\Omega(x)\}$. With these notions in hand, the radial Gauss mapping $\mathscr{A} = \mathscr{A}_\Omega$ and the reverse radial Gauss mapping $\mathscr{A}^* = \mathscr{A}^*_\Omega$ are defined as follows: for any $\omega \subseteq \mathbb{S}^n$,

$$\begin{aligned} \mathscr{A}(\omega) &= \{\nu(\vec{r}(\xi)) : \ \xi \in \omega\}, \\ \mathscr{A}^*(\omega) &= \{\xi \in \mathbb{S}^n : \ \nu(\vec{r}(\xi)) \in \omega\}. \end{aligned} \tag{1.1}$$

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In [33] the L_p dual curvature measures $\widetilde{C}_{p,q}(\Omega,\cdot)$, where $p,q\in\mathbb{R}$, are a two-parameter family of Borel measures on $\mathbb{S}^n$, defined by [1]

$$(1.2)\qquad \widetilde{C}_{p,q}(\Omega,\omega)=\int_{\mathscr{A}^*(\omega)}\frac{r^q(\xi)}{u^p(\mathscr{A}(\xi))}d\sigma_{\mathbb{S}^n}(\xi).$$

The associated Minkowski type problem was posed by Lutwak-Yang-Zhang [33]: Given a finite Borel measure μ on $\mathbb{S}^n$, find necessary and sufficient conditions on μ so that it is the L_p dual curvature measure of a convex body. If μ is absolutely continuous w.r.t. $\sigma_{\mathbb{S}^n}$ and $f=\frac{d\mu}{d\sigma_{\mathbb{S}^n}}$ is the Radon-Nikodym derivative, then, in terms of the support function u, the problem reduces to the following Monge-Ampère equations

$$(1.3)\qquad \det(\nabla^2u+uI)=\left(u^2+|\nabla u|^2\right)^{\frac{n+1-q}{2}}u^{p-1}f(x)\quad\text{on }\mathbb{S}^n,$$

where ∇ is the covariant derivative w.r.t. an orthonormal frame on $\mathbb{S}^n$.

The L_p dual Minkowski problem includes the classical Minkowski problem as a special case, and unifies the L_p-Minkowski problem and dual Minkowski problem introduced in [25, 31]. There is a large number of papers devoted to these problems, see e.g. [4, 7, 16, 17, 19, 21, 30, 32] for the L_p-Minkowski problem, and [5, 15, 24, 25, 29, 38] for the dual Minkowski problem.

The L_p-Minkowski problem amounts to solve (1.3) with $q=n+1$. It is of particular interest, as the problem describes the self-similar solutions to the flows by powers of the Gauss curvature [3, 22]:

$$(1.4)\qquad \partial_tX(x,t)=-K^\alpha(x,t)\nu(x,t),$$

where $X(\cdot,t)$ is a time-dependent embedding of a family of convex hypersurfaces $\mathcal{M}_t$, $K(\cdot,t)$ and $\nu(\cdot,t)$ denote the Gauss curvature and unit outer normal of $\mathcal{M}_t$ respectively. In fact the self-similar solutions to (1.4) satisfy (1.3) with $f\equiv1$ and $p=1-1/\alpha$. For $\alpha=1$, flow (1.4) was first studied by Firey [20] to model the shape change of tumbling stones. It was conjectured that, when $\alpha>1/(n+2)$, flow (1.4) deforms each convex hypersurface in $\mathbb{R}^{n+1}$ into a round point. Andrews proved the conjecture for the case $n=1$ in [2], and for the case $n=2$ and $\alpha=1$ in [1]. Very recently Brendle-Choi-Daskalopoulos [8] resolved this conjecture for all dimensions $n\geq2$. This shows that $u\equiv1$ is the only solution to (1.3) when $q=n+1$, $p\in(-n-1,1)$ and $f\equiv1$. However

[1] Lutwak-Yang-Zhang's L_p dual curvature measure [33] is more general than (1.2), as their definition allows a dependence of a fixed star body Q (i.e. a compact star-shaped set about the origin). If Q is taken as the unit ball $B_1\subseteq\mathbb{R}^{n+1}$, then their conception is formulated by (1.2).

for non-constant f, the L_p-Minkowski problem admits multiple solutions when $p \leq 0$ [23, 27, 28, 37].

For general $p, q \in \mathbb{R}$, the existence of solutions to (1.3) was partially addressed in [6, 12, 14, 26], and the uniqueness of solutions was proved for $p > q$ [26, 33]. The main goal of this paper is to show a non-uniqueness result for the L_p dual Minkowski problem. We say $u \in C^2(\mathbb{S}^n)$ is *uniformly convex* if u is the support function of a convex body whose boundary has uniformly positive principal curvatures. Our main result is the following.

Theorem 1.1. *Let $f \equiv 1$. Then equation (1.3) admits an even, smooth, uniformly convex, positive solution $u \not\equiv 1$, provided that $p, q \in \mathbb{R}$ satisfy one of the following*

(A1) $q - 2n - 2 > p \geq 0$,
(A2) $q > 0$ *and* $-q^* < p < \min\{0, q - 2n - 2\}$, *where* q^* *is given in* (1.9) *below.*
(A3) $p + 2n + 2 < q \leq 0$.

Clearly $u \equiv 1$ is a solution to (1.3) for $f \equiv 1$. Hence our Theorem 1.1 shows that if one of (A1)-(A3) holds, then besides the unit ball B_1 there is another origin-symmetric convex body Ω whose L_p dual curvature measure coincides with the standard spherical measure $\sigma_{\mathbb{S}^n}$. Since (1.5) is not affine-invariant unless $q = -p = n+1$, ellipsoids are not solutions to the problem in general. In [24] the authors showed that, if $f \equiv 1$, $n = 1$, $p = 0$, and q is an even integer no less than 6, then (1.3) has a non-constant solution. Our Theorem 1.1 (A1) extends their result.

Theorem 1.1 follows from Theorems 1.2 & 1.3 below. Both theorems are proved by studying a Monge-Amperè type functional (1.5). For any finite Borel measure σ on $\mathbb{S}^n$ and integrable function g, we use the following convention:

$$\fint_{\mathbb{S}^n} g d\sigma := \frac{1}{\sigma(\mathbb{S}^n)} \int_{\mathbb{S}^n} g d\sigma.$$

Let μ and μ^* be two finite Borel measures on $\mathbb{S}^n$. Consider the functional:

$$\mathcal{J}_{p,q,\mu,\mu^*}(\Omega) = \Phi_{p,\mu}(\Omega) + \Psi_{q,\mu^*}(\Omega), \quad \text{for } \Omega \in \mathcal{K}_0, \tag{1.5}$$

where

$$\Phi_{p,\mu}(\Omega) = \begin{cases} -\dfrac{1}{p} \log \displaystyle\fint_{\mathbb{S}^n} u_\Omega^p(x) d\mu(x), & \text{if } p \neq 0, \\ -\displaystyle\fint_{\mathbb{S}^n} \log u_\Omega(x) d\mu(x), & \text{if } p = 0, \end{cases} \tag{1.6}$$

and

$$
\Psi_{q,\mu^*}(\Omega)=\begin{cases}\dfrac{1}{q}\log\displaystyle\fint_{\mathbb{S}^n} r_\Omega^q(\xi)\,d\mu^*(\xi), & \text{if } q\neq 0,\\[2ex] \displaystyle\fint_{\mathbb{S}^n}\log r_\Omega(\xi)\,d\mu^*(\xi), & \text{if } q=0.\end{cases} \tag{1.7}
$$

Observe that $\mathcal{J}_{p,q,\mu,\mu^*}$ is homogeneous degree zero, namely

$$
\mathcal{J}_{p,q,\mu,\mu^*}(t\Omega)=\mathcal{J}_{p,q,\mu,\mu^*}(\Omega),\quad \forall\ t>0. \tag{1.8}
$$

For convenience, we shall omit sometimes the subscript μ (or μ^*) in (1.5)-(1.7) if μ (or μ^*) is exactly the standard spherical measure. We will see that, up to a rescaling, (1.3) is the Euler equation of functional (1.5) for $d\mu=f\,d\sigma_{\mathbb{S}^n}$ and $d\mu^*=d\sigma_{\mathbb{S}^n}$.

Let $\mathcal{K}_0^e\subset\mathcal{K}_0$ be the set of all origin-symmetric convex bodies. By a Blaschke-Santaló type inequality [13], we are able to use a variational argument to prove Theorem 1.2 below, which shows the existence of origin-symmetric solutions to the L_p dual Minkowski problem.

To our best knowledge, even for the symmetric measures, Theorem 1.2 under condition (B2) gives the first existence result for the problem when $p<0$, $q>0$ and $q\neq n+1$, hence is of particular interest. We point out that, under condition (B1) or (B3), the existence of origin-symmetric solutions was obtained in [12, 26] and in [29] for $p=0$. As this existence result is needed in our main result Theorem 1.1, we still include a proof in this paper for reader's convenience.

Theorem 1.2. *Let $d\mu^*=d\sigma_{\mathbb{S}^n}$, $d\mu=f\,d\sigma_{\mathbb{S}^n}$, f be an even function on $\mathbb{S}^n$, and $1/C\leq f\leq C$ for some constant $C>0$. Assume that $p,q\in\mathbb{R}$ satisfy one of the following*

(B1) $p\geq 0$ *and* $q\geq 0$*;*

(B2) $q>0$ *and* $-q^*<p<0$*, where* $q^*>0$ *is defined as*

$$
q^*=\begin{cases}\dfrac{q}{q-n} & \text{if}\ \ q>n+1,\\ n+1 & \text{if}\ \ q=n+1,\\ \dfrac{nq}{q-1} & \text{if}\ \ 1<q<n+1,\\ +\infty & \text{if}\ \ 0<q\leq 1.\end{cases} \tag{1.9}
$$

(B3) $p\leq 0$ *and* $q\leq 0$*.*

Then there is a convex body $\Omega_0 \in \mathcal{K}_0^e$ such that

$$\mathcal{J}_{p,q,\mu}(\Omega_0) = \max\{\mathcal{J}_{p,q,\mu}(\Omega) : \ \Omega \in \mathcal{K}_0^e\}. \tag{1.10}$$

Moreover $\partial\Omega_0$ is strictly convex and is $C^{1,\gamma}$ for some $\gamma \in (0,1)$, and satisfies

$$\widetilde{C}_{p,q}(\Omega_0, \omega) = \lambda_{\Omega_0} \int_\omega f d\sigma_{\mathbb{S}^n}, \quad \textit{for any Borel set } \omega \subseteq \mathbb{S}^n, \tag{1.11}$$

where

$$\lambda_{\Omega_0} = \frac{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}}{\int_{\mathbb{S}^n} u^p d\mu}. \tag{1.12}$$

If f is additionally smooth, then the support function u_{Ω_0} is an even, smooth, uniformly convex, positive solution to (1.3) with f replaced by $\lambda_{\Omega_0} f$.

For $p \neq q$, $\widetilde{\Omega}_0 := \lambda_{\Omega_0}^{\frac{1}{p-q}} \Omega_0$ solves the L_p dual Minkowski problem, namely $\widetilde{\Omega}_0$ satisfies

$$\widetilde{C}_{p,q}(\widetilde{\Omega}_0, \omega) = \int_\omega f d\sigma_{\mathbb{S}^n}, \quad \textit{for any Borel set } \omega \subseteq \mathbb{S}^n, \tag{1.13}$$

and $u_{\widetilde{\Omega}_0}$ is an even, smooth, uniformly convex, positive solution to (1.3), provided f is additionally smooth.

Remark 1.1. *In [26], the existence of solutions to (1.3) when f is not necessarily even was obtained for $p > q$. When $p \leq q$, the existence result, without evenness assumption on f, becomes much more difficult. It was available for $p > -n-1$ and $q = n+1$ [19], and for $p > 1$ and $q > 0$ [6]. In a subsequent paper [14], we will prove that, for $p > 0$ and all $q \in \mathbb{R}$, the problem admits a weak solution if the prescribed measure μ is not concentrated on any closed hemisphere, while the evenness of μ is not required.*

Remark 1.2. *As in [4], we are able to prove by approximation the L_p dual Minkowski problem admits an origin-symmetric solution when p, q satisfy condition (B2) in Theorem 1.2, and f is an even and nonnegative function on $\mathbb{S}^n$, $\int_{\mathbb{S}^n} f d\sigma_{\mathbb{S}^n} > 0$, and $L^{\frac{q^*}{q^*+p}}$-integrable (when $q^* \neq +\infty$) or L^s-integrable for some $s > 1$ (when $q^* = +\infty$) . See Theorem 3.2 in Section 3 below.*

We then show that, if $\mu = \mu^* = \sigma_{\mathbb{S}^n}$ and $q > p + 2n + 2$, then the unit ball B_1 is not a maximiser of (1.10). This together with Theorem 1.2 proves Theorem 1.1.

Theorem 1.3. *Let $\mu = \mu^* = \sigma_{\mathbb{S}^n}$. If $q > p + 2n + 2$, then there is an even function $\eta \in C^\infty(\mathbb{S}^n)$, and a small $\varepsilon > 0$, such that the convex body $\Omega_t \in \mathcal{K}_0^e$, whose support* *function is $u(x,t)=1+t\eta(x)$, satisfies*

$$\mathcal{J}_{p,q}(B_1)<\mathcal{J}_{p,q}(\Omega_t),\quad \textit{for } t\in(0,\varepsilon). \tag{1.14}$$

This paper is organised as follows. In Section 2, we calculate the first and second variations of functional (1.5). We show that B_1 is an unstable critical point of the functional $\mathcal{J}_{p,q}$ provided $q>p+2n+2$, which consequently proves Theorem 1.3. In Section 3, we prove Theorem 1.2 via variational argument, and then complete the proof of Theorem 1.1. The Poincáre inequality on $\mathbb{S}^n$ is related to the stability of B_1 under the functional (1.5). It can be obtained by studying the eigenvalues of the spherical Laplace operator [36]. In Section 4, we provide an alternative proof for the Poincáre inequality with sharp constant via the uniqueness of the self-similar solution to the flow (1.4) when $\alpha>\frac{1}{n+2}$ [1, 2, 8], which makes our paper self-contained.

2. Second variation for Monge-Ampère type functional (1.5)

Let u and r be respectively the support function and radial function of $\Omega\in\mathcal{K}_0$. Given any $\eta\in C^0(\mathbb{S}^n)$, there is an $\varepsilon>0$, depending on $\min_{\mathbb{S}^n}u$ and $\max_{\mathbb{S}^n}|\eta|$, such that $u(x)+t\eta(x)>0$ for all $x\in\mathbb{S}^n$ and $|t|<\varepsilon$. Consider a family of convex bodies

$$\Omega_t=\{z:\ z\cdot x\le u(x)+t\eta(x),\ x\in\mathbb{S}^n\},\quad \text{for } |t|<\varepsilon. \tag{2.1}$$

Let $u(x,t)$ and $r(x,t)$ be the support function and radial function of Ω_t.

Lemma 2.1. *Suppose that $\partial\Omega$ is C^1 and strictly convex at $z_0\in\partial\Omega$. Then the limits below exist*

$$\dot u(x_0):=\lim_{t\to0}\frac{u(x_0,t)-u(x_0,0)}{t},$$
$$\dot r(\xi_0):=\lim_{t\to0}\frac{r(\xi_0,t)-r(\xi_0,0)}{t},$$

where x_0 is the unit outer normal of $\partial\Omega$ at z_0 and $\xi_0=z_0/|z_0|=\mathscr{A}^*_\Omega(x_0)$. Furthermore

$$\dot u(x_0)=\eta(x_0), \tag{2.2}$$

and

$$\frac{\dot r}{r}(\xi_0)=\frac{\dot u}{u}(x_0). \tag{2.3}$$

Proof. By (2.1) and the definition of support function, we have

$$u(x,t)\le u(x)+t\eta(x),\quad \text{for all } x\in\mathbb{S}^n,\ |t|<\varepsilon. \tag{2.4}$$

Therefore

$$\limsup_{t\to 0^+} \frac{u(x_0,t)-u(x_0,0)}{t} \le \eta(x_0). \tag{2.5}$$

On the other hand, let $u_{z_0}(x) = u(x) - z_0 \cdot x$ and $u_{z_0}(x,t) = u(x,t) - z_0 \cdot x$. Since $\partial\Omega$ is C^1 at z_0, one infers that there exists a $x_t \in \mathbb{S}^n$ so that

$$u_{z_0}(x_0,t) = (u_{z_0} + t\eta)(x_t) \ \text{ with } x_t \to x_0 \text{ as } t \to 0. \tag{2.6}$$

For this, as $u_{z_0}(x_0,0) = u_{z_0}(x_0) = 0$, if $x_{t_k} \to x_1$ then $u_{z_0}(x_1) = 0$ and so x_1 is a unit outer normal at z_0. Hence x_1 must coincide with x_0. Therefore, by $u_{z_0}(x_t) \ge 0$,

$$\begin{aligned} \liminf_{t\to 0^+} \frac{u(x_0,t)-u(x_0,0)}{t} &= \liminf_{t\to 0^+} \frac{u_{z_0}(x_0,t)-u_{z_0}(x_0)}{t} \\ &= \liminf_{t\to 0^+} \frac{u_{z_0}(x_t)+t\eta(x_t)}{t} \\ &\ge \eta(x_0). \end{aligned} \tag{2.7}$$

For $t \to 0^-$, (2.4) and (2.6) give respectively

$$\liminf_{t\to 0^-} \frac{u(x_0,t)-u(x_0,0)}{t} \ge \eta(x_0), \ \text{ and } \limsup_{t\to 0^-} \frac{u(x_0,t)-u(x_0,0)}{t} \le \eta(x_0).$$

Hence (2.2) follows.

Next we prove (2.3). For this, let $h(x) = (\eta/u)(x) \in C^0(\mathbb{S}^n)$. By (2.2), we have

$$u(x_0,t) = u(x_0) + t\eta(x_0) + o(t). \tag{2.8}$$

It follows that

$$\begin{aligned} 0 &= -\log r(\xi_0) + \log u(x_0) - \log(\xi_0 \cdot x_0) \\ &= -\log r(\xi_0) + \log u(x_0,t) - \log(\xi_0 \cdot x_0) + \big(\log u(x_0) - \log u(x_0,t)\big) \\ &\ge -\log r(\xi_0) + \log r(\xi_0,t) - th(x_0) + o(t). \end{aligned} \tag{2.9}$$

On the other hand, since $\partial\Omega$ is strictly convex at z_0, there is a $\xi_t \in \mathbb{S}^n$ such that

$$-\log r(\xi_t,t) = \log(\xi_t \cdot x_0) - \log u(x_0,t) \ \text{ with } \xi_t \to \xi_0 \text{ as } t \to 0.$$

This together with (2.8) shows that

$$\begin{aligned} 0 &\le -\log r(\xi_t) + \log u(x_0) - \log(\xi_t \cdot x_0) \\ &= -\log r(\xi_t) + \log u(x_0) - \big(\log u(x_0,t) - \log r(\xi_t,t)\big) \\ &= -\log r(\xi_0) + \log r(\xi_0,t) - th(x_0) + o(t). \end{aligned} \tag{2.10}$$

We complete the proof by (2.9) and (2.10).

$\square$

In the rest of this section, we always assume that u is uniformly convex. Then the radial Gauss mapping $\mathscr{A}$ and the reverse radial Gauss mapping $\mathscr{A}^*$, defined by (1.1), are one-to-one mappings. Given $\omega \subset \mathbb{S}^n$, we consider the "cone-like" region inside Ω

$$\mathcal{C} := \{z \in \mathbb{R}^{n+1} : \ z = \lambda \nu^{-1}(x), \ \lambda \in [0,1], \ x \in \omega\},$$

where ν denotes the spherical image of Ω. It is well-known that the volume element of $\mathcal{C}$ can be expressed by

$$d\mathrm{Vol}(\mathcal{C}) = \frac{1}{n+1}\frac{u(x)}{K(\nu^{-1}(x))}d\sigma_{\mathbb{S}^n}(x) = \frac{1}{n+1}r^{n+1}(\xi)d\sigma_{\mathbb{S}^n}(\xi),$$

where $K(z)$ is the Gauss curvature of $\partial\Omega$ at z. It follows that the determinants of the Jacobian of the mappings $\mathscr{A}$ and $\mathscr{A}^*$ are given by

$$\begin{aligned} |\mathrm{Jac}\mathscr{A}^*|(x) &= \frac{u(x)}{r^{n+1}(\mathscr{A}^*(x))K(\nu^{-1}(x))}, \\ |\mathrm{Jac}\mathscr{A}|(\xi) &= \frac{r^{n+1}(\xi)K(\vec{r}(\xi))}{u(\mathscr{A}(\xi))}. \end{aligned} \tag{2.11}$$

Let $\eta \in C^\infty(\mathbb{S}^n)$. By the uniform convexity of u, there is a small $\varepsilon > 0$ such that, for all $|t| < \varepsilon$, (i) Ω_t defined by (2.1) lies in $\mathcal{K}_0$; (ii) $u(x,t) := u(x) + t\eta$ is the support function of Ω_t; and (iii) $u(x,t)$ is uniformly convex. Let us compute the first and second variations of functional (1.5).

Proposition 2.1. *Let $\Omega \in \mathcal{K}_0$ be a convex body whose support function u is uniformly convex. Given $\eta \in C^\infty$, let Ω_t be the convex bodies defined by (2.1). Let $d\mu = f d\sigma_{\mathbb{S}^n}$ and $d\mu^* = f^* d\sigma_{\mathbb{S}^n}$. Denote by $\alpha = n+1-q$, $\beta = p-1$. Then*

$$\frac{d}{dt}\Big|_{t=0}\mathcal{J}_{p,q,\mu,\mu^*}(\Omega_t) = \frac{1}{\int_{\mathbb{S}^n} r^q d\mu^*}\int_{\mathbb{S}^n} J_{p,q,\mu,\mu^*}(x)\eta(x)d\sigma_{\mathbb{S}^n}(x), \tag{2.12}$$

where

$$J_{p,q,\mu,\mu^*}(x) = \frac{f^*\circ\mathscr{A}^*_\Omega}{(r\circ\mathscr{A}^*_\Omega)^\alpha K} - \lambda u^\beta f(x), \quad \textit{with } \lambda = \frac{\int_{\mathbb{S}^n} r^q d\mu^*}{\int_{\mathbb{S}^n} u^p d\mu},$$

and K is the Gauss curvature of $\partial\Omega$ calculated at $\nu_\Omega^{-1}(x)$.

If Ω is a convex body satisfying $J_{p,q,\mu,\sigma_{\mathbb{S}^n}} \equiv 0$, then

$$\begin{aligned}
(2.13) \qquad & \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{J}_{p,q,\mu,\sigma_{\mathbb{S}^n}}(\Omega_t) \\
= \quad & \frac{1}{\int_{\mathbb{S}^n} u^p d\mu} \Big\{ \int_{\mathbb{S}^n} \Big(\sum h^{ij}\eta_{ij} + H\eta \Big) u^\beta \eta d\mu - \alpha \int_{\mathbb{S}^n} u^\beta \frac{u\eta + \nabla u \cdot \nabla \eta}{(r \circ \mathscr{A}^*_\Omega)^2} \eta d\mu \\
& -\beta \int_{\mathbb{S}^n} u^{\beta-1}\eta^2 d\mu + \frac{p-q}{\int_{\mathbb{S}^n} u^p d\mu} \Big(\int_{\mathbb{S}^n} u^\beta \eta d\mu \Big)^2 \Big\},
\end{aligned}$$

where $\{h^{ij}\}$ is the inverse matrix of $\{u_{ij} + u\delta_{ij}\}$, and $H = \sum h^{ii}$ is the mean curvature of $\partial\Omega$.

Proof. Note that, for all $|t| < \varepsilon$, $\partial\Omega_t$ are C^2 and strictly convex, with uniformly convex support function $u(x,t) = u(x) + t\eta$. Denote by $r = r(\xi,t)$ the radial function of Ω_t. Hence, by (2.11) and Lemma 2.1, we compute the first variation of (1.5) as follows

$$\begin{aligned}
\frac{d}{dt}\mathcal{J}_{p,q,\mu,\mu^*}(\Omega_t) &= -\frac{1}{⨍_{\mathbb{S}^n} u^p d\mu} ⨍_{\mathbb{S}^n} u^\beta \eta d\mu + \frac{1}{⨍_{\mathbb{S}^n} r^q d\mu^*} ⨍_{\mathbb{S}^n} r^q \frac{\dot r}{r} d\mu^* \\
&= -\frac{1}{\int_{\mathbb{S}^n} u^p d\mu} \int_{\mathbb{S}^n} u^\beta \eta d\mu + \frac{1}{\int_{\mathbb{S}^n} r^q d\mu^*} \int_{\mathbb{S}^n} \frac{f^* \circ \mathscr{A}^*_{\Omega_t}}{(r \circ \mathscr{A}^*_{\Omega_t})^\alpha K} \eta d\sigma_{\mathbb{S}^n} \\
(2.14) \qquad &= \frac{1}{\int_{\mathbb{S}^n} r^q d\mu^*} \Big\{ \int_{\mathbb{S}^n} \frac{f^* \circ \mathscr{A}^*_{\Omega_t}}{(r \circ \mathscr{A}^*_{\Omega_t})^\alpha K} \eta d\sigma_{\mathbb{S}^n} - \frac{\int_{\mathbb{S}^n} r^q d\mu^*}{\int_{\mathbb{S}^n} u^p d\mu} \int_{\mathbb{S}^n} u^\beta f \eta d\sigma_{\mathbb{S}^n} \Big\},
\end{aligned}$$

where the geometric quantities above are of Ω_t. Taking $t = 0$, we get (2.12).

Note that $r \circ \mathscr{A}^*_{\Omega_t} = \sqrt{u^2_{\Omega_t} + |\nabla u_{\Omega_t}|^2}$. Letting $\mu^* = \sigma_{\mathbb{S}^n}$ in (2.14) and then differentiating (2.14) w.r.t. t again, we further calculate, by the assumption $J_{p,q,\mu,\sigma_{\mathbb{S}^n}} \equiv 0$,

$$\begin{aligned}
(2.15) \quad & \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{J}_{p,q,\mu,\sigma_{\mathbb{S}^n}}(\Omega_t) \\
= \quad & \frac{1}{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}} \Big\{ \int_{\mathbb{S}^n} \sum S^{ij}_n (\eta_{ij} + \eta\delta_{ij}) \frac{\eta}{(r \circ \mathscr{A}^*_\Omega)^\alpha} d\sigma_{\mathbb{S}^n} \\
& -\alpha \int_{\mathbb{S}^n} \frac{u\eta + \nabla u \cdot \nabla \eta}{(r \circ \mathscr{A}^*_\Omega)^{\alpha+2} K} \eta d\sigma_{\mathbb{S}^n} - \lambda\beta \int_{\mathbb{S}^n} u^{\beta-1} f \eta^2 d\sigma_{\mathbb{S}^n} \\
& -\frac{q}{\int_{\mathbb{S}^n} u^p d\mu} \Big(\int_{\mathbb{S}^n} \frac{\eta}{(r \circ \mathscr{A}^*_\Omega)^\alpha K} d\sigma_{\mathbb{S}^n} \Big) \Big(\int_{\mathbb{S}^n} u^\beta \eta d\mu \Big) + p \frac{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}}{\Big(\int_{\mathbb{S}^n} u^p d\mu \Big)^2} \Big(\int_{\mathbb{S}^n} u^\beta \eta d\mu \Big)^2 \Big\} \\
= \quad & \frac{1}{\int_{\mathbb{S}^n} u^p d\mu} \Big\{ \int_{\mathbb{S}^n} u^\beta \big(\sum h^{ij}\eta_{ij} + H\eta \big) \eta d\mu - \alpha \int_{\mathbb{S}^n} u^\beta \frac{u\eta + \nabla u \cdot \nabla \eta}{r^2} \eta d\mu \\
& -\beta \int_{\mathbb{S}^n} u^{\beta-1}\eta^2 d\mu + \frac{p-q}{\int_{\mathbb{S}^n} u^p d\mu} \Big(\int_{\mathbb{S}^n} u^\beta \eta d\mu \Big)^2 \Big\}.
\end{aligned}$$

This finishes the proof.

$\square$

By virtue of Proposition 2.1, we are able to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\eta \in C^\infty(\mathbb{S}^n)$ be an even function. As the unit ball B_1 is uniformly convex, there is a small $\varepsilon = \varepsilon_\eta > 0$, depending on η, such that, for all $|t| < \varepsilon$, $\Omega_t^\eta := \{z \in \mathbb{R}^{n+1} : x \cdot z \le 1 + t\eta(x),\ x \in \mathbb{S}^n\}$ has support function $u(x,t) = 1 + t\eta(x)$, which is positive and uniformly convex. Clearly $\Omega_t^\eta \in \mathcal{K}_0^e$.

By Proposition 2.1,

$$\frac{d}{dt}\Big|_{t=0} \mathcal{J}_{p,q}(\Omega_t^\eta) = 0, \tag{2.16}$$

and

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{J}_{p,q}(\Omega_t^\eta) &= \fint_{\mathbb{S}^n} \big(\eta\Delta\eta + (n-\alpha-\beta)\eta^2\big) d\sigma_{\mathbb{S}^n} + (p-q)\Big(\fint_{\mathbb{S}^n} \eta d\sigma_{\mathbb{S}^n}\Big)^2 \\ &= (q-p)\fint_{\mathbb{S}^n} \big(\eta - \bar\eta\big)^2 d\sigma_{\mathbb{S}^n} - \fint_{\mathbb{S}^n} |\nabla\eta|^2 d\sigma_{\mathbb{S}^n}, \end{aligned} \tag{2.17}$$

where $\bar\eta := \fint_{\mathbb{S}^n} \eta d\sigma_{\mathbb{S}^n}$ is the mean value of η.

By (i) in Theorem 4.1 below, there is an $\eta_0 \in C^\infty(\mathbb{S}^n)$, with $\bar\eta_0 = 0$, $\eta_0 \not\equiv 0$, such that

$$(2n + 2 + \frac{1}{2}\delta_{p,q})\fint_{\mathbb{S}^n} \eta_0^2 d\sigma_{\mathbb{S}^n} \ge \fint_{\mathbb{S}^n} |\nabla\eta_0|^2 d\sigma_{\mathbb{S}^n}.$$

where

$$\delta_{p,q} := q - p - 2n - 2 > 0.$$

Then for $\Omega_t := \Omega_t^{\eta_0}$, whose support function is $1 + t\eta_0$, one has by (2.17)

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{J}_{p,q}(\Omega_t) \ge \frac{1}{2}\delta_{p,q}\fint_{\mathbb{S}^n} \eta_0^2 d\sigma_{\mathbb{S}^n} > 0. \tag{2.18}$$

For $\varepsilon_0 = \varepsilon_{\eta_0} > 0$ very small, one knows that $\partial\Omega_t$ is smooth and uniformly convex for all $|t| < \varepsilon_0$. Hence by (2.17) and (2.18)

$$\begin{aligned} \mathcal{J}_{p,q}(\Omega_t) &= \mathcal{J}_{p,q}(B_1) + t\frac{d}{dt}\Big|_{t=0}\mathcal{J}_{p,q}(\Omega_t) + \frac{1}{2}t^2\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{J}_{p,q}(\Omega_t) + o(t^2) \\ &> \mathcal{J}_{p,q}(B_1), \quad \text{for } t \in (0, \varepsilon_0'), \end{aligned}$$

provided ε_0', depending on η_0, is sufficiently small.

$\square$

Remark 2.1. *When $p = 0$ and $q = n+1$, the second variation of the functional was obtained in* [22].

3. Proof of Theorems 1.1 & 1.2

In this section, we first prove Theorem 1.2. This together with Theorem 1.3 shows Theorem 1.1.

Given $\Omega \in \mathcal{K}_0^e$, the polar set of Ω is defined as follows

$$\Omega^* = \{y \in \mathbb{R}^{n+1} : y \cdot x \leq 1 \ \ \forall\, x \in \Omega\}.$$

The following generalised Blaschke-Santaló inequality was proved in [13].

Theorem 3.1 (Blaschke-Santaló type inequality [13])**.** *Given $q > 0$, let $q^* > 0$ be the number given by* (1.9)*. For $\gamma \in (0, q^*]$, $\gamma \neq +\infty$, there is a constant $C_{n,q,\gamma} > 0$ such that,*

$$\left(\fint_{\mathbb{S}^n} r_\Omega^q d\sigma_{\mathbb{S}^n}\right)^{\frac{1}{q}} \left(\fint_{\mathbb{S}^n} r_{\Omega^*}^\gamma d\sigma_{\mathbb{S}^n}\right)^{\frac{1}{\gamma}} \leq C_{n,q,\gamma}, \quad \forall\, \Omega \in \mathcal{K}_0^e. \tag{3.1}$$

Theorem 3.1 enables us to solve the optimisation problem (1.10).

Proposition 3.1. *Under the assumptions of Theorem 1.2, there is a convex body $\Omega_0 \in \mathcal{K}_0^e$ solving the maximisation problem* (1.10)*.*

Proof. We prove Proposition 3.1 when either (B1) or (B2) holds. For the case (B3), we use a dual argument.

Assume that either (B1) or (B2) is satisfied. We have $p \leq 0$ and $q > 0$. By the homogeneity (1.8), it suffices to show there is a $\Omega_0 \in \mathcal{K}_0^e$, with $\Psi_{q,\sigma_{\mathbb{S}^n}}(\Omega_0) = 0$, such that

$$\widetilde{\Phi}_{p,\mu}(\Omega_0) = \max_{\Omega \in \mathcal{K}_0^e} \left\{ \widetilde{\Phi}_{p,\mu}(\Omega) : \ \Psi_{q,\sigma_{\mathbb{S}^n}}(\Omega) = 0 \right\}, \tag{3.2}$$

where

$$\widetilde{\Phi}_{p,\mu}(\Omega) := \begin{cases} -\dfrac{1}{p} \displaystyle\int_{\mathbb{S}^n} u_\Omega^p d\mu, & \text{if } p \neq 0, \\ -\displaystyle\int_{\mathbb{S}^n} \log u_\Omega d\mu, & \text{if } p = 0. \end{cases}$$

For (3.2), let $\{\Omega_j\} \subset \mathcal{K}_0^e$, with $\Psi_{q,\sigma_{\mathbb{S}^n}}(\Omega_j) = 0$, be a maximising sequence. We denote $u_j = u_{\Omega_j}$ and $r_j = r_{\Omega_j}$ for convenience. We claim

$$\max_{\mathbb{S}^n} u_j \leq C, \tag{3.3}$$

for some $C > 0$, independent of j. We next prove (3.3) (under the assumption (B1) or (B2)) case by case: (i) $p < 0$; (ii) $p = 0$; (iii) $p > 0$.

Case I: $p<0$. We follow an argument in [19]. Let $\delta>0$ be a fixed small constant. Set $S_1^j=\mathbb{S}^n\cap\{u_j\le\delta\}$, $S_2^j=\mathbb{S}^n\cap\{\delta<u_j<1/\delta\}$ and $S_3^j=\mathbb{S}^n\cap\{u_j\ge 1/\delta\}$. Since Ω_j is origin-symmetric, we conclude that

$$|S_1^j|\to 0,\ |S_2^j|\to 0,\ \text{as } L_j:=\max_{\mathbb{S}^n} u_j\to\infty. \tag{3.4}$$

Let Ω^* be the polar set of Ω, and $r_j^*=r_{\Omega_j^*}$. It is well known that $r_j^*=1/u_j$, see e.g. [35]. Denote $\gamma=-p>0$. By condition (B2), $\gamma<q^*$. We have

$$\begin{aligned}\widetilde{\Phi}_{p,\mu}(\Omega_j) &= \frac{1}{\gamma}\int_{S_1^j\cup S_2^j\cup S_3^j} u_j^{-\gamma} f d\sigma_{\mathbb{S}^n}\\ &\le C\int_{S_1^j} r_j^{*\gamma} d\sigma_{\mathbb{S}^n}+C_\delta|S_2^j|+C\delta^\gamma\\ &\le C\Big(\int_{\mathbb{S}^n} r_j^{*q'} d\sigma_{\mathbb{S}^n}\Big)^{\frac{\gamma}{q'}}|S_1^j|^{1-\frac{\gamma}{q'}}+C_\delta|S_2^j|+C\delta^\gamma,\end{aligned}$$

for any $\gamma<q'<q^*$. Since $\fint_{\mathbb{S}^n} r_j^q d\sigma_{\mathbb{S}^n}=1$, if $L_j\to\infty$, then by (3.1) and (3.4)

$$\limsup_{j\to\infty}\widetilde{\Phi}_{p,\mu}(\Omega_j)\le C\delta^\gamma.$$

As $\widetilde{\Phi}_{p,\mu}(\Omega_j)\ge\widetilde{\Phi}_{p,\mu}(B_1)>0$, we arrive a contradiction by letting $\delta\to 0$.

Case II: $p=0$. Let $l_j=\min_{\mathbb{S}^n} r_j$ and $L_j=\max_{\mathbb{S}^n} r_j$. By a rotation of coordinates we may assume that $L_j=r_j(e_1)$. Since Ω_j is origin-symmetric, the points $\pm L_je_1\in\partial\Omega_j$. Hence

$$u_j(x)=\max\{z\cdot x:\ x\in\Omega_j\}\ge L_j|x\cdot e_1|,\quad \forall\, x\in\mathbb{S}^n. \tag{3.5}$$

Therefore

$$\begin{aligned}\widetilde{\Phi}_{p,\mu}(\Omega_j) &\le -(\log L_j)/C-\int_{\mathbb{S}^n}\log|x\cdot e_1|f(x)d\sigma_{\mathbb{S}^n}(x)\\ &\le -(\log L_j)/C+C,\end{aligned}$$

which implies that $\widetilde{\Phi}_{p,\mu}(\Omega_j)\to-\infty$, if $L_j\to\infty$. This cannot occur as $\{\Omega_j\}$ is a maximising sequence.

Case III: $p>0$. Again let $L_j=\max_{\mathbb{S}^n} r_j$. As in Case II, we have (3.5). Hence

$$\widetilde{\Phi}_{p,\mu}(\Omega_j)\le-\frac{1}{p}\int_{\{x\in\mathbb{S}^n:\ x\cdot e_1\ge\frac12\}} u_j^p d\mu\le -L_j^p/C\to-\infty\ \text{ if } L_j\to\infty.$$

Since $\{\Omega_j\}$ is a maximising sequence, one infers $L_j\le C$.

Combining Case I-III, we have proved (3.3) under the assumption (B1) or (B2).

Let $w_j^+ = \max_{x\in\mathbb{S}^n}\big(u_j(x)+u_j(-x)\big)$ and $w_j^- = \min_{x\in\mathbb{S}^n}\big(u_j(x)+u_j(-x)\big)$ be the maximum and the minimum of the width of Ω_j. We next show that

$$w_j^- \geq 1/C, \tag{3.6}$$

for some $C>0$, independent of j. This estimate together with (3.3) means that Ω_j is of uniformly good shape.

For $0<q<n+1$, we have

$$1 = \Big(\fint_{\mathbb{S}^n} r_j^q d\sigma_{\mathbb{S}^n}\Big)^{\frac{n+1}{q}} \leq \fint_{\mathbb{S}^n} r_j^{n+1} d\sigma_{\mathbb{S}^n} \leq C\text{Volume}(\Omega_j) \leq (w_j^+)^n w_j^-,$$

which shows (3.6) by using (3.3).

For $q\geq n+1$, we have

$$1 = \fint_{\mathbb{S}^n} r_j^q d\sigma_{\mathbb{S}^n} = (w_j^+)^q \fint_{\mathbb{S}^n}\Big(\frac{r_j}{w_j^+}\Big)^q d\sigma_{\mathbb{S}^n} \leq C(w_j^+)^{q-n-1}\text{Volume}(\Omega_j) \leq C(w_j^+)^{q-1}w_j^-.$$

Again, (3.6) follows from (3.3).

As above, $l_j = \min_{\mathbb{S}^n} r_j$. Assume without loss of generality that $l_j = r_j(e_1)$. By the symmetry of Ω_j,

$$l_j \geq r_j(\xi)|\xi\cdot e_1|, \quad \forall\, \xi\in\mathbb{S}^n.$$

For $q=0$, we thus have

$$0 = \fint_{\mathbb{S}^n} \log r_j d\sigma_{\mathbb{S}^n} \leq \log l_j - \fint_{\mathbb{S}^n} \log|\xi\cdot e_1| d\sigma_{\mathbb{S}^n} \leq \log l_j + C.$$

This shows that $l_j\geq\delta$ for some $\delta>0$ uniformly, and so (3.6) follows.

In virtue of (3.3) and (3.6), we conclude by the Blaschke selection theorem that Ω_j, after passing to a subsequence, converges to a $\Omega_0\in\mathcal{K}_0^e$ in Hausdorff distance, thus completing the proof under the assumption (B1) or (B2).

For case (B3), let $\{\Omega_j\}\subset\mathcal{K}_0^e$ be a maximising sequence of functional $\mathcal{J}_{p,q,\mu}$. Let $p'=-q$ and $q'=-p$, and Ω_j^* be the polar set of Ω_j. One easily sees that

$$\mathcal{J}_{p,q,\mu,\sigma_{\mathbb{S}^n}}(\Omega) = \mathcal{J}_{p',q',\sigma_{\mathbb{S}^n},\mu}(\Omega^*) \ \ \forall\,\Omega\in\mathcal{K}_0^e. \tag{3.7}$$

It then follows that $\{\Omega_j^*\}$ is a maximising sequence of $\mathcal{J}_{p',q',\sigma_{\mathbb{S}^n},\mu}$. Observe that if p,q satisfy (B3), then p',q' satisfy (B1). Hence, by our previous argument for (B1), it is not hard to conclude that, after a proper rescaling, $t_j\Omega_j^*$ converges to a $\Omega_0^*\in\mathcal{K}_0^e$ such that

$$\mathcal{J}_{p',q',\sigma_{\mathbb{S}^n},\mu}(\Omega_0^*) = \max\{\mathcal{J}_{p',q',\sigma_{\mathbb{S}^n},\mu}(\Omega):\ \Omega\in\mathcal{K}_0^e\}$$

By (3.7), we conclude that $\Omega_0 = \Omega_0^*$ satisfies (1.10). Note that if p', q' satisfy (B2), then p, q also satisfy (B2). Hence we would not get more by applying the above dual argument to (B2).

□

We then show that, after a dilation, the maximiser Ω_0 obtained in Proposition 3.1 is a solution to the L_p dual Minkowski problem, under an additional assumption: $\partial\Omega_0$ is C^1 and strictly convex.

Proposition 3.2. *If $\partial\Omega_0$ is C^1 and strictly convex, then Ω_0 satisfies* (1.11)*.*

Proof. Let u and r be respectively the support function and radial function of Ω_0. For any even function $\eta \in C^0(\mathbb{S}^n)$, let $\Omega_t \in \mathcal{K}_0^e$ be the convex bodies given by (2.1), with u replaced by u_0. Denote by $u(x,t)$ and $r(x,t)$ the support function and radial function of Ω_t. By Lemma 2.1, we have as in proof of Proposition 2.1

$$\frac{d}{dt}\Big|_{t=0}\mathcal{J}_{p,q,\mu}(\Omega_t) = \frac{1}{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}}\Big(-\lambda_{\Omega_0}\int_{\mathbb{S}^n} u^{p-1}\eta d\mu + \int_{\mathbb{S}^n}\frac{r^q}{u\circ\mathscr{A}_{\Omega_0}}\eta\circ\mathscr{A}_{\Omega_0} d\sigma_{\mathbb{S}^n}\Big).$$

By [33, Lemma 5.1], we further calculate

$$\frac{d}{dt}\Big|_{t=0}\mathcal{J}_{p,q,\mu}(\Omega_t) = \frac{1}{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}}\Big(-\lambda_{\Omega_0}\int_{\mathbb{S}^n} u^{p-1}\eta d\mu + \int_{\mathbb{S}^n} u^{p-1}\eta d\widetilde{C}_{p,q}(\Omega_0,\cdot)\Big). \tag{3.8}$$

Since Ω_0 is the maximiser and η is arbitrary, we deduce that

$$\int_{\mathbb{S}^n} g d\widetilde{C}_{p,q}(\Omega_0,\cdot) = \lambda_{\Omega_0}\int_{\mathbb{S}^n} g d\mu, \quad \forall \text{ even function } g \in C^0(\mathbb{S}^n),$$

thus completing the proof by the evenness of f.

□

Proposition 3.3. *Let Ω_0 be the maximiser obtained in Proposition 3.1. Then $\partial\Omega_0$ is strictly convex and is $C^{1,\gamma}$ for some $\gamma \in (0,1)$*

Proof. Let u be the support function of Ω_0 and $\bar{u} = \bar{u}_{\Omega_0}$ be its homogeneous degree one extension, namely $\bar{u} : \mathbb{R}^{n+1} \to \mathbb{R}$, defined by

$$\bar{u}(Y) = \sup_{Z\in\Omega_0} Y\cdot Z.$$

The face of Ω_0 with outer normal $Y \in \mathbb{R}^{n+1}$ is then given by

$$F_{\Omega_0}(Y) = \{Z \in \Omega_0 : \ \bar{u}(Y) = Y\cdot Z\},$$

which lies in $\partial\Omega_0$ provided $Y \neq 0$, and

$$\partial\bar{u}(Y) = F_{\Omega_0}(Y), \tag{3.9}$$

where $\partial\bar{u}(Y) := \{X \in \mathbb{R}^{n+1} : \bar{u}(Z) \geq \bar{u}(Y) + \langle X, Z - Y\rangle,\ \forall Z \in \mathbb{R}^{n+1}\}$ is the subgradient of $\bar{u}$ at Y. See Schneider's book [35] for all this.

For $\mathbf{e} \in \mathbb{S}^n$, let $L_{\mathbf{e}}$ be the hyperplane in $\mathbb{R}^{n+1}$ which is tangential to $\mathbb{S}^n$ at $\mathbf{e}$. Denote by $\pi = \pi_{\mathbf{e}} : \mathbb{R}^n \to \mathbb{S}^n$ the radial projection from $L_{\mathbf{e}}$ to $\mathbb{S}^n$,

$$\pi(y) = \frac{y + \mathbf{e}}{\sqrt{1 + |y|^2}}.$$

Let $v = v_{\mathbf{e}} : \mathbb{R}^n \to \mathbb{R}$ be the restriction of $\bar{u}$ on $L_{\mathbf{e}}$, that is

$$v(y) = \bar{u}(y + \mathbf{e}) = \sqrt{1 + |y|^2}\, u(\pi(y)). \tag{3.10}$$

It is not hard to check by (3.9) and (3.10) that

$$\partial v(y) = \big\{X - (X \cdot \mathbf{e})\mathbf{e} : X \in \partial\bar{u}(y + \mathbf{e})\big\}. \tag{3.11}$$

Let $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure. Recall that the surface area measure $\mathcal{S}(\Omega_0, \cdot)$ is defined as

$$\mathcal{S}(\Omega_0, \omega) = \mathcal{H}^n(\nu_{\Omega_0}^{-1}(\omega)), \quad \text{for Borel set } \omega \subset \mathbb{S}^n. \tag{3.12}$$

It follows from (3.9)-(3.11) that for any $D \subset \mathbb{R}^n$

$$\mathscr{M}_v(D) = \int_{\pi(D)} \langle x, \mathbf{e}\rangle \ d\mathcal{S}(\Omega_0, x), \tag{3.13}$$

where $\mathscr{M}_v(D) := \mathcal{H}^n(\partial v(D))$ is the Monge-Ampère measure associated to v. We claim that, $\mathcal{S}(\Omega_0, \cdot)$ is absolutely continuous w.r.t. $\sigma_{\mathbb{S}^n}$, and there is a $C > 0$, such that

$$1/C \leq \varrho_{\Omega_0} := \frac{d\mathcal{S}(\Omega_0, \cdot)}{d\sigma_{\mathbb{S}^n}} \leq C. \tag{3.14}$$

Note that ϱ_{Ω_0} is the reciprocal Gauss curvature if Ω_0 is C^2 smooth. Once (3.14) is proved, we deduce by (3.13) and $\det D\pi(y) = (1 + |y|^2)^{-\frac{n+1}{2}}$ that

$$d\mathscr{M}_v = \frac{\varrho_{\Omega_0} \circ \pi}{(1 + |y|^2)^{\frac{n+2}{2}}} dy. \tag{3.15}$$

For (3.15), one may consult [18, 35] for a full discussion. By (3.14) and (3.15), the density of the Monge-Ampère measure of v in a compact set is bounded between two constants. For a given $y_0 \in \mathbb{R}^n$, let ℓ_{y_0} be the support function of $v(y)$ at y_0. In view of (3.11), the contact set $\mathcal{C}_{y_0} := \{y \in \mathbb{R}^n : v(y) = \ell_{y_0}(y)\}$ cannot contain a straight line in $\mathbb{R}^n$. Hence we conclude by [9, 11] that v is strictly convex and $C^{1,\gamma'}_{\mathrm{loc}}$ for some

$\gamma' \in (0,1)$. See also [19]. This implies that $\partial\Omega_0$ is strictly convex. Let $\varphi : D' \to \mathbb{R}$ be the convex function such that $\{(x, \varphi(x)) : x \in D'\} \subseteq \partial\Omega_0$, where D' is a closed convex domain, containing the origin, lying in $\Omega_0 \cap \{X \in \mathbb{R}^{n+1} : X \cdot \mathbf{e} = 0\}$. One can check that φ is exactly the Legendre transform of v. Therefore, by(3.14) and (3.15), $d\mathscr{M}_\varphi/dx$ is bounded between two positive constants. By [9, 11], φ is $C^{1,\gamma}$ for some $\gamma \in (0,1)$.

It remains to show (3.14). For $\eta \in C^0(\mathbb{S}^n)$, consider

$$\Omega_t = \{z \in \mathbb{R}^{n+1} : \ x \cdot z \leq e^{t\eta(x)} u(x), \ \forall \ x \in \mathbb{S}^n\},$$

which is a perturbation of Ω_0. Denote by $u^t = u(x,t)$ and $r^t = r(\xi, t)$ the support and radial functions of Ω_t. It follows from [33, Theorem 6.4] that

$$\frac{d}{dt}\Big|_{t=0} \Psi_q(r^t) = \frac{1}{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}} \int_{\mathbb{S}^n} \eta d\widetilde{C}_q(\Omega_0, \cdot). \tag{3.16}$$

Since $u^t \leq e^{t\eta} u$, one has

$$\lim_{t \to 0^+} \frac{u^t(x) - u(x)}{t} \leq \eta u(x). \tag{3.17}$$

By (3.16) and (3.17), we obtain

$$\begin{aligned} 0 \ &\geq \ \lim_{t\to 0^+} \frac{\mathcal{J}_{p,q}(\Omega_t) - \mathcal{J}_{p,q}(\Omega_0)}{t} \\ &\geq \ \frac{1}{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}} \Big\{ -\lambda_{\Omega_0} \int_{\mathbb{S}^n} u^p \eta d\mu + \int_{\mathbb{S}^n} \eta d\widetilde{C}_q(\Omega_0, \cdot) \Big\} \\ &= \ \frac{1}{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}} \Big\{ -\lambda_{\Omega_0} \int_{\mathbb{S}^n} u^p f \eta d\sigma_{\mathbb{S}^n} + \int_{\mathbb{S}^n} (r \circ \mathscr{A}^*_{\Omega_0})^{q-n-1} u \eta d\mathcal{S}(\Omega_0, \cdot) \Big\}, \end{aligned}$$

where λ_{Ω_0} is given by (1.12), and the last equality is due to [25, Lemma 3.7]. Since η is arbitrary, u and r are bounded between two positive constants, we get

$$\varrho_{\Omega_0} \leq C.$$

Let Ω_0^* be the polar set of Ω_0. Then $r^* = 1/u$, see e.g. [35]. For $\eta \in C^0(\mathbb{S}^n)$, consider

$$\Omega_t^* = \mathrm{conv}\{e^{t\eta(x)} r^*(x) x : \ x \in \mathbb{S}^n\},$$

and $\Omega_t = (\Omega_t^*)^*$. Denote by $u^{*t} = u^*(\xi, t)$ and $r^{*t} = r^*(x,t)$ the support and radial function of Ω_t^*, by $u^t = u(x,t)$ and $r^t = r(\xi,t)$ the support and radial function of Ω_t. Since $r^{*t} \geq e^{t\eta} r^*$, one gets $u^t \leq e^{-t\eta} u$. Therefore

$$\lim_{t\to 0^+} \frac{u^t(x) - u(x)}{t} \leq -\eta u(x). \tag{3.18}$$

By [33, Theorem 6.1],

$$\left.\frac{d}{dt}\right|_{t=0}\Psi_q(r^t) = -\frac{1}{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}}\int_{\mathbb{S}^n}\eta d\widetilde{C}_q(\Omega_0,\cdot). \tag{3.19}$$

It follows by (3.18) and (3.19)

$$\begin{aligned} 0 \;&\geq\; \lim_{t\to 0^+}\frac{\mathcal{J}_{p,q}(\Omega_t)-\mathcal{J}_{p,q}(\Omega_0)}{t} \\ &\geq\; \frac{1}{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}}\Big\{\lambda_{\Omega_0}\int_{\mathbb{S}^n} u^p f\eta d\sigma_{\mathbb{S}^n} - \int_{\mathbb{S}^n}(r\circ\mathscr{A}^*_{\Omega_0})^{q-n-1}u\eta d\mathcal{S}(\Omega_0,\cdot)\Big\}, \end{aligned}$$

which shows that

$$\varrho_{\Omega_0}\geq 1/C.$$

This completes the proof.

□

Remark 3.1. *To see the maximiser of the optimisation problem* (1.10) *is a solution to the L_p dual Minkowski problem* (1.11)*, one can also follow the argument in* [25, Lemma 5.1].

We are at the position to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. By Proposition 3.1-3.3, it remains to show $u = u_{\Omega_0}$ is smooth and uniformly convex. Note that for $p\neq q$, it is not hard to see $\widetilde{\Omega}_0 := \lambda_{\Omega_0}^{\frac{1}{p-q}}\Omega_0$ satisfies (1.13), and $u_{\widetilde{\Omega}_0}$ solves (1.3). By the homogeneity (1.8), $\widetilde{\Omega}_0$ is also a maximiser for (1.10).

By [25, Lemma 3.7] and [33, Proposition 5.4], it follows from (3.8) that, $\forall\,\eta\in C^0(\mathbb{S}^n)$,

$$\left.\frac{d}{dt}\right|_{t=0}\mathcal{J}_{p,q,\mu}(\Omega_t) = \frac{1}{\int_{\mathbb{S}^n} r^q d\sigma_{\mathbb{S}^n}}\Big(-\lambda_{\Omega_0}\int_{\mathbb{S}^n} u^{p-1}\eta f d\sigma_{\mathbb{S}^n} + \int_{\mathbb{S}^n}(r\circ\mathscr{A}^*_{\Omega_0})^{q-n-1}\eta d\mathcal{S}(\Omega_0,\cdot)\Big).$$

Since Ω_0 is the maximiser of (1.10), we obtain

$$\frac{d\mathcal{S}(\Omega_0,\cdot)}{d\sigma_{\mathbb{S}^n}} = \lambda_{\Omega_0}(r\circ\mathscr{A}^*_{\Omega_0})^{n+1-q}u^{p-1}f. \tag{3.20}$$

Given any $\mathbf{e}\in\mathbb{S}^n$, let v and $\varphi = v^*$ (the Legendre transform of v) be as in Proposition 3.3. Then

$$\det D^2 v = \lambda_{\Omega_0}(1+|y|^2)^{-\frac{n+1+p}{2}}v^{p-1}(|Dv|^2+(Dv\cdot y-v)^2)^{\frac{n+1-q}{2}}f\circ\pi, \tag{3.21}$$

and

$$\det D^2\varphi = \lambda_{\Omega_0}^{-1}(1+|D\varphi|^2)^{\frac{n+1+p}{2}}(D\varphi\cdot x-\varphi)^{1-p}(|x|^2+\varphi^2)^{-\frac{n+1-q}{2}}/f\Big(\frac{D\varphi,-1}{\sqrt{1+|D\varphi|^2}}\Big), \tag{3.22}$$

in the Aleksandrov sense. By Proposition 3.3, v and φ are strictly convex and are $C^{1,\gamma}$ for some $\gamma \in (0,1)$.

If f is Hölder, then the right hand sides of (3.21) and (3.22) are both Hölder continuous. By [10], v and φ are both $C^{2,\gamma'}$ for some $\gamma' \in (0,1)$. Smoothness of v and $v\phi$ then follows from the standard theory of uniformly elliptic equations, provided $f \in C^\infty(\mathbb{S}^n)$. Hence $\partial\Omega_0$ is smooth. By (3.20), $u \in C^\infty(\mathbb{S}^n)$ solves (1.3) with f replaced by $\lambda_{\Omega_0} f$.

□

The following result improves Theorem 1.2 under condition (B2).

Theorem 3.2. *Let p, q satisfy condition (B2) in Theorem 1.2, and $d\mu = f d\sigma_{\mathbb{S}^n}$, where f is an even, non-negative function and $\int_{\mathbb{S}^n} f d\sigma_{\mathbb{S}^n} > 0$. Assume that $f \in L^{\frac{q^*}{q^*+p}}(\mathbb{S}^n)$ if $q^* \neq +\infty$, or $f \in L^s(\mathbb{S}^n)$ for some $s > 1$ if $q^* = +\infty$. Then there is a convex body $\Omega \in \mathcal{K}_0^e$ such that $\widetilde{C}_{p,q}(\Omega, \omega) = \mu(\omega)$ for all Borel set $\omega \subseteq \mathbb{S}^n$.*

Proof. We use an approximation argument similar to [4]. For positive integers j, let $d\mu_j = f_j d\sigma_{\mathbb{S}^n}$ be a sequence of measures, where f_j is a truncation of f,

$$f_j(x) = \begin{cases} j & \text{if } f(x) \geq j, \\ f(x) & \text{if } 1/j < f(x) < j, \\ 1/j & \text{if } f(x) \leq 1/j. \end{cases}$$

Recall that $\mathcal{J}_{p,q,\mu_j}$ satisfies (1.8). Hence by Theorem 1.2, there is a $\widetilde{\Omega}_j \in \mathcal{K}_0^e$ such that,

$$\widetilde{C}_{p,q}(\widetilde{\Omega}_j, \omega) = \mu_j(\omega), \text{ for any Borel set } \omega \subseteq \mathbb{S}^n, \tag{3.23}$$

and if $\tilde{r}_j = r_{\widetilde{\Omega}_j}$ and $\tilde{u}_j = u_{\widetilde{\Omega}_j}$ then

$$\Big(\fint_{\mathbb{S}^n} \tilde{u}_j^p d\mu_j\Big)^{-\frac{1}{p}}\Big(\fint_{\mathbb{S}^n} \tilde{r}_j^q d\sigma_{\mathbb{S}^n}\Big)^{\frac{1}{q}} = \exp \mathcal{J}_{p,q,\mu_j}(\widetilde{\Omega}_j) \geq \exp \mathcal{J}_{p,q,\mu_j}(B_1) \geq 1/C_{f,n,p}, \tag{3.24}$$

for a positive constant $C_{f,n,p} > 0$, independent of j.

Let $\Omega_j = \lambda_j \widetilde{\Omega}_j$, where $\lambda_j = \Big(\fint_{\mathbb{S}^n} \tilde{r}_j^q d\sigma_{\mathbb{S}^n}\Big)^{-\frac{1}{q}}$ so that

$$\fint_{\mathbb{S}^n} r_{\Omega_j}^q d\sigma_{\mathbb{S}^n} = 1. \tag{3.25}$$

Let $u_j = u_{\Omega_j}$, $r_j = r_{\Omega_j}$ and $L_j := \max_{\mathbb{S}^n} u_j$. As in the proof of Proposition 3.1, for a small constant $\delta > 0$, let $S_1^j = \mathbb{S}^n \cap \{u_j \le \delta\}$, $S_2^j = \mathbb{S}^n \cap \{\delta < u_j < 1/\delta\}$ and $S_3^j = \mathbb{S}^n \cap \{u_j \ge 1/\delta\}$. It is not hard to see that

$$|S_1^j| \to 0 \text{ and } |S_2^j| \to 0, \quad \text{if } L_j \to \infty. \tag{3.26}$$

First let us consider the case $q^* \neq \infty$. Denote $\gamma = -p > 0$ and $r_j^* = r_{\Omega_j^*}$, the radial function of Ω_j^* (the polar set of Ω_j). We have

$$\begin{aligned}
\int_{\mathbb{S}^n} u_j^p d\mu_j &= \int_{S_1^j \cup S_2^j \cup S_3^j} u_j^{-\gamma} f_j d\sigma_{\mathbb{S}^n} \\
&\le C\Big(\int_{\mathbb{S}^n} r_j^{*q^*} d\sigma_{\mathbb{S}^n}\Big)^{\frac{\gamma}{q^*}} \Big(\int_{S_1^j} f_j^{\frac{q^*}{q^*-\gamma}} d\sigma_{\mathbb{S}^n}\Big)^{\frac{q^*-\gamma}{q^*}} + C_\delta \int_{S_2^j} f_j d\sigma_{\mathbb{S}^n} + C\delta^\gamma \qquad (3.27)\\
&\to C\delta^\gamma, \quad \text{if } L_j \to \infty, \qquad (3.28)
\end{aligned}$$

where $f \in L^{\frac{q^*}{q^*+p}}(\mathbb{S}^n)$, (3.1), (3.25) and (3.26) are used for the last line. As the LHS of (3.24) is rescaling invariant, its value is unchanged if $\tilde{u}_j, \tilde{r}_j$ are replaced by u_j, r_j. We conclude that L_j are uniformly bounded, by (3.25), (3.28) and sending $\delta \to 0$. As in the proof of Proposition 3.1, we also deduce from (3.25) that $l_j := \min_{\mathbb{S}^n} u_j$ stay uniformly away from zero.

In view of (3.23), for $q^* \neq +\infty$.

$$\int_{\mathbb{S}^n} \tilde{r}_j^q d\sigma_{\mathbb{S}^n} = \int_{\mathbb{S}^n} \tilde{u}_j^p d\mu_j = \int_{\mathbb{S}^n} \tilde{r}_j^{*\gamma} f_j d\sigma_{\mathbb{S}^n} \le \Big(\int_{\mathbb{S}^n} \tilde{r}_j^{*q^*} d\sigma_{\mathbb{S}^n}\Big)^{\frac{\gamma}{q^*}} \Big(\int_{\mathbb{S}^n} f_j^{\frac{q^*}{q^*-\gamma}} d\sigma_{\mathbb{S}^n}\Big)^{\frac{q^*-\gamma}{q^*}}. \tag{3.29}$$

The first equality in (3.29) together with (3.24) shows that

$$\int_{\mathbb{S}^n} \tilde{r}_j^q d\sigma_{\mathbb{S}^n} \ge 1/C_{f,n,p,q} > 0.$$

While the inequality in (3.29), (3.1) and $f \in L^{\frac{q^*}{q^*+p}}(\mathbb{S}^n)$ give

$$\int_{\mathbb{S}^n} \tilde{r}_j^q d\sigma_{\mathbb{S}^n} \le C_{f,n,p,q}.$$

Hence $1/C_{f,n,p,q} \le \lambda_j \le C_{f,n,p,q}$, for a constant $C_{f,n,p,q} > 0$ only depending on f, n, p, q.

The above estimates for L_j, l_j, λ_j imply $\max_{\mathbb{S}^n} u_{\widetilde{\Omega}_j}$ and $\min_{\mathbb{S}^n} u_{\widetilde{\Omega}_j}$ are uniformly bounded from above and below. By the Blaschke selection theorem, $\widetilde{\Omega}_j$ converges, after passing to a subsequence, to a $\Omega \in \mathcal{K}_0^e$ in Hausdorff distance. By the weak convergence of L_p dual curvature measures [33, Proposition 5.2], it follows from (3.23) that $\widetilde{C}_{p,q}(\Omega, \omega) = \mu(\omega)$, thus completing the proof for $q^* \neq +\infty$.

When $q^* = +\infty$, (3.27) and (3.29) still hold if q^* is replaced by $\alpha = \frac{s\gamma}{s-1}$. Hence we can finish the proof by the same discussion as above.

$\square$

Next we prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 1.2 and by the homogeneity (1.8), there is a $\Omega_0 \in \mathcal{K}_0^e$ such that u_{Ω_0} is a solution to the equation (1.3) with $f \equiv 1$ and $\mathcal{J}_{p,q}(\Omega_0) = \max\{\mathcal{J}_{p,q}(\Omega) : \Omega \in \mathcal{K}_0^e\}$. We deduce from Theorem 1.3 that

$$\mathcal{J}_{p,q}(B_1) < \mathcal{J}_{p,q}(\Omega_0).$$

Therefore $u_{\Omega_0} \neq u_{B_1}$. While $u_{B_1} \equiv 1$ and u_{Ω_0} both solve (1.3) when $f \equiv 1$.

$\square$

4. Sharp Poincaré inequality on $\mathbb{S}^n$

This section is devoted to the Poincaré inequality on $\mathbb{S}^n$. This inequality is well-known and has many applications. It can be proved by studying the eigenvalues of the spherical Laplace operator [36]. We prove it by the stability of the unit ball B_1 under the functional (1.5) and the uniqueness of the self-similar solution to the powered Gauss curvature flow (1.4).

Theorem 4.1. *We have*

(i)

$$\inf\left\{\frac{⨍_{\mathbb{S}^n} |\nabla\eta|^2 d\sigma_{\mathbb{S}^n}}{⨍_{\mathbb{S}^n} \eta^2 d\sigma_{\mathbb{S}^n}} : \ \eta \in C^\infty(\mathbb{S}^n) \text{ is even}, ⨍_{\mathbb{S}^n} \eta d\sigma_{\mathbb{S}^n} = 0, \eta \not\equiv 0 \right\} = 2n+2;$$

(ii)

$$\inf\left\{\frac{⨍_{\mathbb{S}^n} |\nabla\eta|^2 d\sigma_{\mathbb{S}^n}}{⨍_{\mathbb{S}^n} \eta^2 d\sigma_{\mathbb{S}^n}} : \eta \in C^\infty(\mathbb{S}^n),\ ⨍_{\mathbb{S}^n} \eta d\sigma_{\mathbb{S}^n} = 0, \eta \not\equiv 0 \right\} = n.$$

Remark 4.1. *By approximation, Theorem 4.1 holds for $\eta \in W^{1,2}(\mathbb{S}^n)$.*

Proof of (i) in Theorem 4.1. Let $\Omega_t = \{z \in \mathbb{R}^{n+1} : z \cdot x \leq 1 + t\eta(x)\} \in \mathcal{K}_0^e$. Consider

$$\mathcal{J}_p(\Omega_t) := \mathcal{J}_{p,n+1}(\Omega_t).$$

By Proposition 2.1, or more precisely by letting $q = n+1$ in (2.17), we have

$$(4.1)\qquad \frac{d^2}{d^2}\Big|_{t=0}\mathcal{J}_p(\Omega_t) = (n+1-p)\fint_{\mathbb{S}^n}(\eta-\bar{\eta})^2 d\sigma_{\mathbb{S}^n} - \fint_{\mathbb{S}^n}|\nabla\eta|^2 d\sigma_{\mathbb{S}^n}.$$

When $f \equiv 1$ and $q = n+1$, (1.3) is the equation of the self-similar solutions to the flow (1.4) with $p = 1 - 1/\alpha$. By [1, 2, 8], $u \equiv 1$ is the only solution for $p \in (-n-1, 1)$ [2]. By [3, 19], the equation (1.3) with $p > -n-1$ and $q = n+1$ admits a solution which maximises the functional $\mathcal{J}_p$. We therefore conclude that

$$\frac{d^2}{d^2}\Big|_{t=0}\mathcal{J}_p(\Omega_t) \le 0,\ \forall\ p > -n-1.$$

This together with (4.1) implies, by letting $p \to -n-1$,

$$(4.2)\qquad \fint_{\mathbb{S}^n}(\eta-\bar{\eta})^2 d\sigma_{\mathbb{S}^n} \le \frac{1}{2n+2}\fint_{\mathbb{S}^n}|\nabla\eta|^2 d\sigma_{\mathbb{S}^n}.$$

We next show (4.2) is sharp. Assume not, then for sufficiently small $\varepsilon > 0$, there is an even $\eta_\varepsilon \not\equiv 0$, $\bar{\eta}_\varepsilon = 0$, such that

$$\fint_{\mathbb{S}^n}|\nabla\eta_\varepsilon|^2 d\sigma_{\mathbb{S}^n} = (2n+2-2\varepsilon)\fint_{\mathbb{S}^n}\eta_\varepsilon^2 d\sigma_{\mathbb{S}^n}.$$

Let $\Omega_t^{\eta_\varepsilon} = \{z \in \mathbb{R}^{n+1} :\ z\cdot x \le 1 + t\eta_\varepsilon\}$. Then for $p = -n-1+\varepsilon$ we have by (4.1),

$$\frac{d^2}{d^2}\Big|_{t=0}\mathcal{J}_{-n-1+\varepsilon}(\Omega_t^{\eta_\varepsilon}) = \varepsilon\fint_{\mathbb{S}^n}\eta_\varepsilon^2 d\sigma_{\mathbb{S}^n} > 0.$$

This means, by virtue of [3, 19], there is another convex body $\Omega' \ne B_1$ maximising $\mathcal{J}_{-n-1+\varepsilon}$ among $\mathcal{K}_0^e$, and $u_{\Omega'}$ solves (1.3) with $f \equiv 1$, $p = -n-1+\varepsilon$, and $q = n+1$, contradicting with the uniqueness of the solution.

$\square$

Proof of (ii) in Theorem 4.1. Consider the functional

$$(4.3)\qquad \widetilde{\mathcal{J}}_p(\Omega, z) = -\frac{1}{p}\log\fint_{\mathbb{S}^n}u_z^p d\sigma_{\mathbb{S}^n} + \frac{1}{n+1}\log\fint_{\mathbb{S}^n}r_z^{n+1}d\sigma_{\mathbb{S}^n},$$

where $z \in \mathrm{Int}\ \Omega$ and u_z, r_z are the support and radial function of Ω w.r.t. the centre z, namely $u_z(x) = \max\{(y-z)\cdot x : y \in \Omega\}$ and $r_z(\xi) = \max\{\lambda : \lambda\xi + z \in \Omega\}$. This functional was used by Andrews-Guan-Ni [3] in the study of the flow (1.4) with $\alpha = (1-p)^{-1}$. Note that the second term on the RHS of (4.3) is independent of z, as $\frac{1}{n+1}\int_{\mathbb{S}^n} r_z^{n+1}d\sigma_{\mathbb{S}^n} = \mathrm{Volume}(\Omega)$.

[2]In fact $u \equiv 1$ is also the unique solution for $p > 1$ and $p \ne n+1$, see e.g. [32].

Given Ω, let $z_e = z_e(\Omega)$ be the entropy point of Ω, namely z_e minimises

$$z \mapsto \widetilde{\mathcal{J}}_p(\Omega, z), \text{ among all } z \in \text{Int } \Omega.$$

For $p < 1$, it was proved in [3] that for each bounded convex Ω with Int $\Omega \neq \emptyset$, there exists a unique entropy point $z_e \in \text{Int}\Omega$, and one readily sees

$$\int_{\mathbb{S}^n} \frac{x}{u_{z_e}^{1-p}(x)} d\sigma_{\mathbb{S}^n}(x) = 0. \tag{4.4}$$

Let $\Omega_t = \{z \in \mathbb{R}^{n+1} : z \cdot x \leq 1 + t\eta\} \in \mathcal{K}_0$. Denote by $z(t) = z_e(\Omega_t)$, the entropy point of Ω_t. By Lemma 2.1, we compute, for $|t|$ very small,

$$\frac{d}{dt}\widetilde{\mathcal{J}}_p(\Omega_t, z(t)) = \frac{1}{\int_{\mathbb{S}^n} r_z^{n+1} d\sigma_{\mathbb{S}^n}} \Big(\int_{\mathbb{S}^n} \frac{\eta}{K} d\sigma_{\mathbb{S}^n} - \frac{\int_{\mathbb{S}^n} r_z^{n+1} d\sigma_{\mathbb{S}^n}}{\int_{\mathbb{S}^n} u_z^p d\sigma_{\mathbb{S}^n}} \int_{\mathbb{S}^n} u_z^{p-1}(\eta - \dot{z} \cdot x) d\sigma_{\mathbb{S}^n} \Big), \tag{4.5}$$

where u_z, r_z are support and radial function of Ω_t w.r.t. $z = z(t)$, and K is the Gauss curvature of Ω_t. Differentiating (4.5) again, we obtain

$$\begin{aligned} &\frac{d^2}{dt^2}\widetilde{\mathcal{J}}_p(\Omega_t, z(t)) \\ = \ & \frac{1}{\int_{\mathbb{S}^n} r_z^{n+1} d\sigma_{\mathbb{S}^n}} \Big\{ \int_{\mathbb{S}^n} \frac{h^{ij}(\eta_{ij} + \eta\delta_{ij})}{K} \eta d\sigma_{\mathbb{S}^n} - \beta \frac{\int_{\mathbb{S}^n} r_z^{n+1} d\sigma_{\mathbb{S}^n}}{\int_{\mathbb{S}^n} u_z^p d\sigma_{\mathbb{S}^n}} \int_{\mathbb{S}^n} u_z^{p-2} \eta(\eta - \dot{z} \cdot x) d\sigma_{\mathbb{S}^n} \\ & - \frac{n+1}{\int_{\mathbb{S}^n} u_z^p d\sigma_{\mathbb{S}^n}} \int_{\mathbb{S}^n} \frac{\eta}{K} d\sigma_{\mathbb{S}^n} \int_{\mathbb{S}^n} u_z^{p-1} \eta d\sigma_{\mathbb{S}^n} + p \frac{\int_{\mathbb{S}^n} r_z^{n+1} d\sigma_{\mathbb{S}^n}}{\Big(\int_{\mathbb{S}^n} u_z^p d\sigma_{\mathbb{S}^n}\Big)^2} \Big(\int_{\mathbb{S}^n} u_z^{p-1} \eta d\sigma_{\mathbb{S}^n} \Big)^2 \Big\}, \end{aligned} \tag{4.6}$$

where $\beta = p - 1$ and h^{ij} is the inverse matrix of $u_{ij} + u\delta_{ij}$. By (4.4), one has

$$\int_{\mathbb{S}^n} u_z^{p-1} \dot{z} \cdot x d\sigma_{\mathbb{S}^n} = 0, \quad \forall\ t.$$

Differentiate this identity w.r.t. t to get

$$\int_{\mathbb{S}^n} u_z^{p-2} \eta \dot{z} \cdot x d\sigma_{\mathbb{S}^n} = \int_{\mathbb{S}^n} u_z^{p-2} (\dot{z} \cdot x)^2 d\sigma_{\mathbb{S}^n}. \tag{4.7}$$

Hence one infers by plugging (4.7) in (4.6) and by $\Omega_0 = B_1$, $z(0) = 0$,

$$\begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} \widetilde{\mathcal{J}}_p(\Omega_t, z(t)) &= \fint_{\mathbb{S}^n} (\Delta\eta + n\eta)\eta d\sigma_{\mathbb{S}^n} - \beta \fint_{\mathbb{S}^n} \eta^2 d\sigma_{\mathbb{S}^n} + \beta \fint_{\mathbb{S}^n} (\dot{z} \cdot x)\eta d\sigma_{\mathbb{S}^n} \\ & \quad -(n+1-p) \Big(\fint_{\mathbb{S}^n} \eta d\sigma_{\mathbb{S}^n} \Big)^2 \\ &= -\fint_{\mathbb{S}^n} |\nabla\eta|^2 d\sigma_{\mathbb{S}^n} + (n+1-p) \fint_{\mathbb{S}^n} (\eta - \bar{\eta})^2 d\sigma_{\mathbb{S}^n} + \beta \fint_{\mathbb{S}^n} (\dot{z} \cdot x)^2 d\sigma_{\mathbb{S}^n}. \end{aligned} \tag{4.8}$$

It is straightforward to see

$$(4.9)\qquad \fint_{\mathbb{S}^n}(\dot z\cdot x)^2 d\sigma_{\mathbb{S}^n} = |\dot z|^2\fint_{\mathbb{S}^n}(\frac{\dot z}{|\dot z|}\cdot x)^2 d\sigma_{\mathbb{S}^n} = |\dot z|^2\fint_{\mathbb{S}^n} x_1^2 d\sigma_{\mathbb{S}^n}.$$

Since $z(t)$ is determined by (4.4), hence depends on p. For clarification, let us denote it by $z_p(t)$. Then we have by differentiating (4.4)

$$0 = (p-1)\fint_{\mathbb{S}^n} u_{z_p(t)}^{p-2}(\eta - \dot z_p\cdot x)x d\sigma_{\mathbb{S}^n},\ \text{for all } p<1.$$

Sending $t=0$, we have

$$\fint_{\mathbb{S}^n}\eta x d\sigma_{\mathbb{S}^n} = \fint_{\mathbb{S}^n}(\dot z_p(0)\cdot x)x d\sigma_{\mathbb{S}^n} = |\dot z_p(0)|\fint_{\mathbb{S}^n}(e_p\cdot x)x d\sigma_{\mathbb{S}^n},$$

where $e_p = \dot z_p(0)/|\dot z_p(0)|$. Multiplying e_p at both sides one deduces

$$(4.10)\qquad |\dot z_p(0)| = \Big(\fint_{\mathbb{S}^n} x_1^2 d\sigma_{\mathbb{S}^n}\Big)^{-1}\fint_{\mathbb{S}^n}(x\cdot e_p)\eta d\sigma_{\mathbb{S}^n} \le C|\eta|_{L^1(\mathbb{S}^n)},\ \text{for all } p<1.$$

As B_1 maximises $\widetilde{\mathcal{J}}_p$ among all $\Omega\in\mathcal{K}_0$ [1, 2, 3, 8], we have by plugging (4.9) in (4.8)

$$0 \ \ge\ -\fint_{\mathbb{S}^n}|\nabla\eta|^2 d\sigma_{\mathbb{S}^n} + (n+1-p)\fint_{\mathbb{S}^n}(\eta-\bar\eta)^2 d\sigma_{\mathbb{S}^n} + (p-1)|\dot z_p(0)|^2\fint_{\mathbb{S}^n} x_1^2 d\sigma_{\mathbb{S}^n}.$$

Sending $p\to 1$, we get by (4.10)

$$(4.11)\qquad \fint_{\mathbb{S}^n}(\eta-\bar\eta)^2 d\sigma_{\mathbb{S}^n} \le \frac{1}{n}\fint_{\mathbb{S}^n}|\nabla\eta|^2 d\sigma_{\mathbb{S}^n}.$$

It remains to show (4.11) is sharp. If not, then for sufficiently small $\varepsilon>0$, there is an $\eta_\varepsilon\not\equiv 0$, $\bar\eta_\varepsilon = 0$, such that

$$\fint_{\mathbb{S}^n}|\nabla\eta_\varepsilon|^2 d\sigma_{\mathbb{S}^n} = (n-\sqrt{\varepsilon})\fint_{\mathbb{S}^n}\eta_\varepsilon^2 d\sigma_{\mathbb{S}^n}.$$

Let $\Omega_t^{\eta_\varepsilon} = \{z\in\mathbb{R}^{n+1}:\ z\cdot x\le 1+t\eta_\varepsilon\}$. Then for $p = 1-\varepsilon$, by (4.8) and (4.10)

$$\begin{aligned}\frac{d^2}{d^2}\Big|_{t=0}\widetilde{\mathcal{J}}_{1-\varepsilon}(\Omega_t^{\eta_\varepsilon}, z(t)) &= (\sqrt{\varepsilon}+\varepsilon)\fint_{\mathbb{S}^n}\eta_\varepsilon^2 d\sigma_{\mathbb{S}^n} - \varepsilon|\dot z_{1-\varepsilon}(0)|^2\fint_{\mathbb{S}^n} x_1^2 d\sigma_{\mathbb{S}^n}\\ &\ge C^{-1}\sqrt{\varepsilon}|\eta_\varepsilon|^2_{L^1(\mathbb{S}^n)} - C\varepsilon|\eta_\varepsilon|^2_{L^1(\mathbb{S}^n)}\\ &> 0,\end{aligned}$$

provided ε sufficiently small. This implies that B_1 is not a maximiser of $\widetilde{\mathcal{J}}_p$, thus arriving a contradiction.

$\square$

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